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Multidimensional Analogue of Virasoro Algebra and Quantum Higher-Order Water Wave Equation

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The bi-Hamiltonian structure of higher-order water wave equation is seen to give rise to a multidimensional analogue of the Virasoro algebra. The quantum version of the equation can be constructed from the corresponding operator product expansion. Possible forms of some conserved quantities are given.

1. INTRODUCTION

Kupershmidt and Mathieu (1989) showed that a quantum version of the KdV and Boussinesq equations can be formulated through the operator product expansion implied by the Virasoro algebra associated with the second Hamiltonian structure. Subsequently it was observed by Roy Chowdhury and Guha (1990) that a perturbed version of the KdV equation and a supersymmetric KdV equation can also be treated in the same fashion. While the underlying algebra in the KdV case is the Virasoro algebra, that in the case of Boussinesq equation is the W_3 algebra. So the equation naturally arises whether there exists any other infinite-dimensional algebra which can help to formulate the quantum version of other integrable systems. In this connection we observe that an extended form of Virasoro algebra exists associated to the second Hamiltonian structure of the higher-order water wave equation whose special case is the Boussinesq equation. While the infinite-dimensional algebra for the Boussinesq case is not a Lie algebra, that occurring in the present case is a Lie algebra.

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2. FORMULATION

The equations under consideration are

$$\begin{aligned} u_t + (uv)_x + \tfrac{1}{3}v_{xx} &= 0 \\ v_t + u_x + vv_c &= 0 \end{aligned} \tag{1}$$

Purkait and Roy Chowdhury (1990) obtained the bi-Hamiltonian structure of the set (1) using a new technique of Fourier transform and small-amplitude expansion. The second symplectic operator is written as

$$M = \frac{4\sqrt{3}}{9}\begin{pmatrix} \frac{1}{3}\partial^3 + \frac{1}{2}(\partial u + u\partial) & \frac{1}{2}v\partial \\ \frac{1}{2}\partial v & \partial \end{pmatrix} \tag{2}$$

that is,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = M \begin{pmatrix} \delta\sigma_2/\delta u \\ \delta\sigma_2/\delta v \end{pmatrix} \tag{3}$$

where σ_2 is given as

$$\sigma_2 = \int \left\{ -\tfrac{3}{2}v_{xx} + \tfrac{3}{2}u_x + \tfrac{3}{4}vv_x - \frac{3\sqrt{3}}{4}\, uv \right\} \tag{4}$$

It is then quite apparent that the Poisson bracket is defined via

$$\{f, g\} = \int dx_1 \langle \nabla f, M\nabla g \rangle \tag{5}$$

where f and g are two functionals of the field variables. Equation (5) entails

$$\begin{aligned} \{u(x), u(x')\} &= \frac{\pi}{c}\,\frac{4\sqrt{3}}{9}\,\{\tfrac{1}{3}\delta'''(x-x') + u\delta'(x-x') + \tfrac{1}{2}u'\delta(x'-x)\} \\ \{u(x), v(x')\} &= 6\pi\,\frac{4\sqrt{3}}{9}\,\frac{v}{2}\,\delta'(x-x') \\ &= \frac{4\pi}{3}\, v\delta'(x-x') \\ \{v(x), v(x')\} &= \frac{4\pi}{3}\,\delta'(x-x') \end{aligned} \tag{6}$$

We now Fourier decompose the fields u and v and write

$$u(x) = \frac{6}{c} \sum_{n=-\alpha}^{\infty} L_n e^{inx} - \tfrac{1}{4}$$
$$v(x) = \frac{6}{c} \sum_{n=-\alpha}^{\infty} S_n e^{inx} - \tfrac{1}{4} \tag{7}$$

whence equations (6) lead immediately to

$$i\{Ln, Lm\} = \frac{\sqrt{3}}{9}\left\{(n-m)Ln+m - \frac{c}{12}(\tfrac{4}{3}\eta^3 + \eta)\delta_{n+m,0}\right\}$$
$$\{Sn, Sm\} = \frac{\eta}{3}\delta_{n+m,0} \tag{8}$$
$$i\{Ln, Sm\} = -\frac{\sqrt{3}}{9}\left(2mSm+n+\frac{\eta}{12}\delta_{n+m,0}\right)$$

which is a simple generalization of the Virasoro algebra that usually occurs in the kdV case. But it is not a supersymmetric one, as there is no anticommuting variables. So we can think of this as a higher-dimensional analogue of the usual Virasoro algebra. Such an extension is also possible for other coupled nonlinear systems (Oevel and Mathieu, 1991), but, so far as the present authors knowledge goes, nothing more is known about them. It is furthermore proved by Kaup (1975) that the inverse problem for (1) is solved by a Schrödinger equation with energy-dependent potential written as

$$\Psi_{xx} + (u + \lambda v - \lambda^2)\Psi = 0 \tag{9}$$

from the reparametrization condition $x \to \sigma(x)$, where σ Diff S'. One can ascertain that $(u + av)$ behaves as a spin-2 field that is the energy momentum tensor $T(x)$. On the other hand, from the bracket relations (8) we can infer that v has spin, and so does λ. So we now proceed to quantize the system with this relevant information. Let us denote the quantum fields pertaining to u by $T(x)$ and that for v by $v(x)$. Now we have (Zamoldochikov *et al.*, 1984)

$$T(z)T(w) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{z-\omega} + \frac{T'(\omega)}{z-\omega}$$
$$T(z)V(\omega) = \frac{V(\omega)}{(z-\omega)^2} + \frac{V(\omega)}{z-\omega} + \frac{1}{(z-\omega)^3} \tag{10}$$
$$V(z)V(\omega) = \frac{1}{(z-\omega)^2} + O(1)\cdots$$

giving the basic operator product expansions. Now consider

$$\dot{T}=[T,P] \tag{11}$$

with

$$\rho=\oint TV\,dz$$

whence

$$\dot{T}=-\oint T(z)\{TV\}(\omega)\,d\omega \tag{12}$$

Now invoking the definition of the normal ordering

$$:(AB)(z):=\oint_z \frac{dx}{(x-z)}A(x)B(z)$$

we get

$$\begin{aligned}
T(z)(TV)(\omega) &= \oint \frac{dx}{x-\omega}-\left\{\underbrace{T(z)T(x)}V(\omega)+T(x)\underbrace{T(z)V(\omega)}\right\} \\
&= \oint \frac{c/2V(\omega)}{(x-\omega)(z-x)^4}\,dx+\oint \frac{2T(x)V(\omega)}{(x-\omega)(z-x)^2}\,dx \\
&\quad +\oint \frac{dx}{x-\omega}\,\frac{\partial x(T(x)V(\omega))}{z-x}+\frac{(TV)(\omega)}{(z-\omega)^2} \\
&= \frac{5}{(z-\omega)^5}+\frac{(4+c/2)V(\omega)}{(z-\omega)^4}+\frac{3V'(\omega)}{(z-\omega)^3} \\
&\quad +\frac{3(TV)(\omega)}{(z-\omega)^2}+\frac{(TV)'(\omega)}{(z-\omega)}\cdots
\end{aligned} \tag{13}$$

So inserting (12) in equation (11), we get

$$\dot{T}=\tfrac{1}{6}(5-c/2)V'''(z)-2(TV)'(z) \tag{14}$$

On the other hand, for the second equation we set

$$\begin{aligned}
\dot{V}&=[V,P] \\
P&=\oint (TV)(z)\,dz
\end{aligned} \tag{15}$$

Using the same procedure as above, we obtain

$$V(z)(TV)(\omega)=\frac{3}{(z-\omega)^4}+\frac{T(\omega)}{(z-\omega)^2}+\frac{(VV)(\omega)}{(z-\omega)^2}$$

which immediately leads to

$$V=-[T'(z)+(VV)'(z)] \tag{16}$$

3. DISCUSSION

We have shown that with the help of the second Hamiltonian structure of the higher-order water wave equation it is possible to define an analogue of the higher-dimensional Virasoro algebra and by using the OPE approach it is possible to quantize it. In this connection we may note that the conserved quantities can be written as,

$$H_1=\oint v\,dz; \qquad H_2=\oint T\,dz; \qquad H_3=\oint (TV)\,dz \tag{17}$$

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